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Orthogonal Polynomials: Their Growth Relative to Their Sums*

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Assume that $a_n \to a/2$ and $b_n \to b$. Let the polynomials $\{p_n\}$ be defined by the recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n = 0, 1, 2, ...,$$

where $p_0 = \text{const} > 0$ and $p_{-1} = 0$. It is proved that for every $0 < p < \infty$

$$\lim_{n\to\infty} \frac{|p_n(x)|^p}{\sum_{k=0}^{n-1} |p_k(x)|^p} = 0 \qquad \text{uniformly for} \quad x \in [b-a, b+a].$$

The uniform convergence actually holds not only when the initial condition p_0 and p_{-1} are restricted to polynomials but also for almost all kinds of initial functions p_0 and p_{-1}. A number of applications to orthogonal polynomials are also studied.

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1. Introduction

Let $\{a_n, b_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that $a_n \neq 0$ for $n > 0$. We define a sequence of functions $\{p_n\}_{n=-1}^{\infty}$ by the recurrence formula

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n = 0, 1, 2, ..., \tag{1.1}$$

where p_0 and p_{-1} are given initial functions. In general, p_n is not a polynomial but if we set

$$p_0 = \text{const} > 0 \qquad \text{and} \qquad p_{-1} = 0, \tag{1.2}$$

then p_n is, in fact, a polynomial of precise degree n. Given a positive Borel measure α on the real line whose moments are finite and whose support is an infinite set, the system of orthogonal polynomials $\{p_n(\alpha)\}_{n=0}^{\infty}$ is defined by the orthogonality relation

$$\int_{\mathbb{R}} p_n(\alpha, t)\, p_m(\alpha, t)\, d\alpha(t) = \delta_{nm}, \qquad n, m \geqslant 0, \tag{1.3}$$

and they satisfy the three-term recurrence (1.1) with the recurrence coefficients

$$\begin{aligned} a_n &= a_n(\alpha) = \int_{\mathbb{R}} tp_n(\alpha, t)\, p_{n-1}(\alpha, t)\, d\alpha(t), \\ b_n &= b_n(\alpha) = \int_{\mathbb{R}} tp_n^2(\alpha, t)\, d\alpha(t), \end{aligned} \tag{1.4}$$

and the initial condition (1.2).

Following the notation in [10],[1] we define the class $M(b, a)$, where $a \geqslant 0$ and $b \in \mathbb{R}$, by

$$M(b, a) = \{\{a_n, b_n\}_{n=0}^{\infty}: a_n > 0, b_n \in \mathbb{R}, \lim_{n\to\infty} a_n = a/2 \text{ and } \lim_{n\to\infty} b_n = b\}. \tag{1.5}$$

We also say that the measure α is in $M(b, a)$ if the corresponding recurrence coefficients $\{a_n(\alpha), b_n(\alpha)\}_{n=0}^{\infty} \in M(b, a)$. By a theorem of Blumenthal (cf. [1; 10, p. 23]), if $\alpha \in M(b, a)$ with $a > 0$, then the support of the measure α satisfies $\operatorname{supp}(\alpha) = [b-a, b+a] \cup S$, where S is bounded and countable with only possible accumulation points in $\{b \pm a\}$. It is well

[1] We point out that there is a slight difference between the notations in [10] and subsequent works, including the present one; namely the parameters a and b in [10] have subsequently been renamed to b and a, respectively.

known that for every positive Borel measure α on the real line and for every real x the inequality

$$\sum_{k=0}^{\infty} p_k^2(\alpha, x) \leqslant \frac{1}{\alpha(\{x\})}, \tag{1.6}$$

holds.[2] Hence, if x is a mass-point of α then $p_n(\alpha, x) \to 0$ as $n \to \infty$, so that for every $0 < p < \infty$ we have

$$\lim_{n\to\infty} \frac{|p_n(\alpha, x)|^p}{\sum_{k=0}^{n-1} |p_k(\alpha, x)|^p} = 0, \qquad x \in \operatorname{supp}(\alpha) \backslash [b-a, b+a], \tag{1.7}$$

whenever $\alpha \in M(b, a)$. It was proved in [10, p. 11, Theorem 3.1.9] that if $\alpha \in M(b, a)$ with $a > 0$, then

$$\lim_{n\to\infty} \max_{x \in [b-a, b+a]} [a^2 - (x-b)^2] \frac{p_n^2(\alpha, x)}{\sum_{k=0}^{n-1} p_k^2(\alpha, x)} = 0. \tag{1.8}$$

Moreover, it has been conjectured that the uniform convergence in (1.8) remains true even if the factor $(a^2 - (x-b)^2)$ is dropped. This is known to be true for Jacobi polynomials (see [10, p. 83]), where $a_n = \frac{1}{2} + O(n^{-2})$ and $b_n = O(n^{-2})$ (see [17, p. 71] or [2, p. 153]).

The primary purpose of our paper is to prove this conjecture in a general setting by allowing complex valued recurrence coefficients in (1.1), by changing the squares to arbitrary positive powers p, and by relaxing (1.2) to arbitrary finite initial value conditions. Furthermore, in case of orthogonal polynomials the uniform convergence is to be proved true on $\operatorname{supp}(\alpha) = [b-a, b+a] \cup S$, which could differ from $[b-a, b+a]$ by countably many points. The main theorems are stated in Section 2, and they are proved in Section 3. Some related applications are discussed in Section 4.

2. Uniform Convergence Theorems

Similarly to (1.5), we define $CM(b, a)$ by

$$CM(b, a) = \{\{a_n, b_n\}_{n=0}^{\infty} : a_n\ (\neq 0) \in \mathbb{C},\ b_n \in \mathbb{C},\ \lim_{n\to\infty} a_n = a/2 \text{ and } \lim_{n\to\infty} b_n = b\}. \tag{2.1}$$

[2] If the moment problem has a unique solution then the sum of the series in (1.6) equals $[\alpha(\{x\})]^{-1}$ (see, e.g., [3, Sect. II.2, p. 25]).

In what follows, for complex a and b we define the complex interval $[b-a, b+a]$ as the line segment connecting the points $b-a$ and $b+a$, that is,

$$[b-a, b+a] \equiv \{b+ta\colon -1 \leqslant t \leqslant 1\}.$$

By Poincaré's theorem [13], $\lim_{n\to\infty} p_{n+1}(x)/p_n(x)$ exists for all x outside the complex interval $[b-a, b+a]$ whenever $\{a_n, b_n\}_{n=0}^{\infty} \in CM(b, a)$ with $a, b \in \mathbb{C}$ and $a \neq 0$, and the limit equals to one of the roots $z_{\pm}$ of the characteristic equation

$$2xz = az^2 + 2bz + a.$$

On the other hand, $\lim_{n\to\infty} p_{n+1}(x)/p_n(x)$ does not need to exist for $x \in [b-a, b+a]$. For instance, if $a>0$, $b \in \mathbb{R}$, $p_{-1}=0$, and $p_0=1$ then $\{p_n\}_{n=1}^{\infty}$ is an orthogonal polynomial system whose zeros are dense in $[b-a, b+a]$ (cf. [1, 9, 10]), and, therefore, the above limits do not exist for $x \in (b-a, b+a)$. The main result of our paper is the following

THEOREM 2.1. *Let $\{a_n, b_n\}_{n=0}^{\infty} \in CM(b, a)$ with $a, b \in \mathbb{C}$, $a \neq 0$, and let the sequence of functions $\{p_n\}_{n=1}^{\infty}$ be generated by* (1.1). *Then for every $0<p<\infty$*

(1) *if the initial functions p_0 and p_{-1} are finite in $[b-a, b+a]$, and $Z=\{x \in \mathbb{R}\colon |p_0(x)|+|p_{-1}(x)|=0\}$, then*

$$\lim_{n\to\infty} \sup_{x \in [b-a, b+a]\setminus Z} \frac{|p_n(x)|^p}{\sum_{k=0}^{n-1} |p_k(x)|^p} = 0 \tag{2.2}$$

(*if the initial functions p_0 and p_{-1} are polynomials then we can take $Z=\varnothing$ in* (2.2)).

(2) *if $\{p_n(\alpha)\}_{n=0}^{\infty}$ is an orthogonal polynomial system associated with a positive measure $\alpha \in M(b, a)$, where $a>0$ and $b \in \mathbb{R}$, then*

$$\lim_{n\to\infty} \max_{x \in \operatorname{supp}(\alpha)} \frac{|p_n(\alpha, x)|^p}{\sum_{k=0}^{n-1} |p_k(\alpha, x)|^p} = 0. \tag{2.3}$$

The proof will be given in Section 3. Here we show why $Z=\varnothing$ when p_0 and p_{-1} are polynomials. The set Z defined in Theorem 2.1 is the set of points x for which $p_n(x)=0$ for every $n = -1, 0, 1, 2, \ldots$. It is quite natural to remove Z because otherwise in (2.2) the division would be impossible. We can take $Z=\varnothing$ as long as we require $0<|p_0(x)|+|p_{-1}(x)|<\infty$ for $x \in [b-a, b+a]$. In the case when p_0 and p_{-1} are polynomials, we let d be the greatest common divisor of p_0 and p_{-1}. Then, d divides all p_n for $n \geqslant -1$ because of the three-term recurrence (1.1). Hence all common zero factors can be cancelled in (2.2) and we can therefore assume that p_0 and p_{-1} have no common zero factors, that is, $Z=\varnothing$. If we have orthogonal polynomials, then $Z=\varnothing$ because $p_0=\text{const}>0$. The difficulty in proving the second part of Theorem 2.1 is that $\operatorname{supp}(\alpha)\backslash[b-a, b+a]$ may not be finite, as a matter of fact, some of the most interesting and most challenging cases, such as certain Pollaczek measures, have infinitely many mass-points outside the interval $[b-a, b+a]$.

Since $a\neq 0$ in Theorem 2.1, we can map the complex interval $[b-a, b+a]$ into the interval $[-1, 1]$ by $\hat{x}=(x-b)/a$. This map gives rise to a new set of functions $\{\hat{p}_n\}_{n=0}^{\infty}$, where $\hat{p}_n(\hat{x})=p_n(x)$. The system $\{\hat{p}_n\}_{n=0}^{\infty}$ satisfies (1.1) with a new set of recursion coefficients $\{\hat{a}_n=a_n/a, \hat{b}_n=(b_n-b)/a\}_{n=0}^{\infty}$, where $\lim_{n\to\infty}\hat{a}_n=\frac{1}{2}$ and $\lim_{n\to\infty}\hat{b}_n=0$, that is, $\{\hat{a}_n, \hat{b}_n\}_{n=0}^{\infty}\in CM(0, 1)$. Therefore, we can always assume that $a=1$ and $b=0$ as long as $a\neq 0$. Hence, instead of (2.2) in Theorem 2.1, we only need to prove

$$\lim_{n\to\infty}\sup_{x\in[-1,1]\backslash Z}\frac{|p_n(x)|^p}{\sum_{k=0}^{n-1}|p_k(x)|^p}=0 \tag{2.4}$$

for $\{a_n, b_n\}_{n=0}^{\infty}\in CM(0, 1)$.

Remark. The unit circle analogue of (2.3) with $p=2$ was first investigated by L. Ya. Geronimus who in [4, p. 38, Theorem 3.4, and p. 40 formula (3.31)] proved a pointwise (but not uniform) version of it under the condition that the corresponding measure is in the Szegő class, that is, the logarithm of the absolutely continuous component of the measure is integrable. Interestingly, Geronimus never emphasizes the pointwise (but not uniform) nature of his result, and, thus, a superfluous examination of his results may give the impression that his estimates are uniform. He even included such an estimate in [4, p. 198, No. I, Table II] with no reference to the proof on [4, pp. 38–40]. One of the innocent victims of this was [15, p. 574, paragraph directly following formula (4.3)] which refers to [4, p. 40, Theorem 3.5] in the (false) belief that it provides uniform estimates for orthogonal polynomials on the unit circle. The complete unit circle version of (2.3) with $p=2$ was proved in [8, p. 55, Theorem 4]. Since the Szegő recursion formula for orthogonal polynomials on the unit circle has a somewhat simpler underlying structure than the three-term recurrence (1.1), the unit circle version of (2.3) with $p=2$ was much easier to prove than (2.3).

In view of Poincaré's theorem [13] on the asymptotic behavior of the ratios f_{n+1}/f_n of solutions $\{f_n\}$ of linear difference equations with variable coefficients, one can easily prove analogues of Theorem 2.1 for x outside the complex interval $[b-a, b+a]$. Surprisingly, the latter can provide a complete characterization of the class $CM(b, a)$. Here we will only prove a particularly elegant version of such characterization theorems which parallels some results in [10, p. 32, Theorem 4.1.12].

THEOREM 2.2. *Let* $\{p_n(\alpha)\}_{n=0}^{\infty}$ *be an orthogonal polynomial system associated with a positive measure* α *supported on a compact set of the real line. Let* $0 < p < \infty$, $a > 0$, *and* $b \in \mathbb{R}$. *Then* $\alpha \in M(b, a)$ *if and only if*

$$\lim_{n \to \infty} \frac{|p_n(\alpha, x)|^p}{\sum_{k=0}^{n-1} |p_k(\alpha, x)|^p} = \left| \frac{x-b}{a} + \sqrt{\left(\frac{x-b}{a}\right)^2 - 1} \right|^p - 1 \tag{2.5}$$

holds uniformly for every compact set $K \subset \mathbb{C} \backslash \operatorname{supp}(\alpha)$. *When p is a positive integer,* (2.5) *remains true with the absolute value signs dropped on both sides.*

3. THE PROOFS

In this section, we will demonstrate Theorems 2.1 and 2.2. As we already remarked in Section 2, it is enough to consider Theorem 2.1 for the case $CM(0, 1)$. In fact, as will be seen from the next lemma, the problem can further be reduced from $a_n \to \frac{1}{2}$ and $b_n \to 0$ as $n \to \infty$ to the extremal case, that is, to $a_n \equiv \frac{1}{2}$ and $b_n \equiv 0$.

LEMMA 3.1. *Let* $0 < p < \infty$. *Suppose* $\{p_k\}_{k=0}^{\infty}$ *is generated by* (1.1) *with recurrence coefficients* $\{a_n, b_n\}_{n=0}^{\infty}$ *and finite initial function* p_0 *and* p_{-1}. *Let* $Z = \{x \in \mathbb{R} : |p_{-1}(x)| + |p_0(x)| = 0\}$. *If* $\{a_n, b_n\}_{n=0}^{\infty} \in CM(0, 1)$, *then for every* $K \subset \mathbb{R}$ *and every positive integer L we have*

$$\varlimsup_{n \to \infty} \sup_{x \in K \backslash Z} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \leqslant \sup_{|c|^p + |s|^p = 1, x \in K} \frac{|R_L(c, s, x)|^p}{\sum_{k=-1}^{L} |R_k(c, s, x)|^p}, \tag{3.1}$$

where $\{R_k(c, s, \cdot)\}_{k=0}^{\infty}$ *is the polynomial system generated by the following recurrence relation*

$$\begin{gathered} xR_n(c, s, x) = \tfrac{1}{2} R_{n+1}(c, s, x) + \tfrac{1}{2} R_{n-1}(c, s, x), \\ R_{-1}(c, s, x) = c, \qquad R_0(c, s, x) = s. \end{gathered} \tag{3.2}$$

Proof. Let $L \geqslant 1$ be fixed, and let $n > L$. By definition, $x \notin Z$ if and only if $|p_{-1}(x)| + |p_0(x)| \neq 0$. In view of the three-term recurrence relation in (1.1), this means that $|p_{k-1}(x)| + |p_k(x)| \neq 0$ for all $k = 0, 1, \ldots$. Hence,

$$\sup_{x \in K \setminus Z} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \leqslant \sup_{x \in K \setminus Z} \frac{|p_n(x)|^p}{\sum_{k=n-L-1}^{n} |p_k(x)|^p}$$

$$= \sup_{x \in K \setminus Z} \frac{|R_{L,n}(p_{n-L-1}(x), p_{n-L}(x), x)|^p}{\sum_{k=-1}^{L} |R_{k,n}(p_{n-L-1}(x), p_{n-L}(x), x)|^p}$$

$$\leqslant \sup_{|c|+|s| \neq 0, x \in K} \frac{|R_{L,n}(c, s, x)|^p}{\sum_{k=-1}^{L} |R_{k,n}(c, s, x)|^p}$$

$$= \sup_{|c|^p+|s|^p=1, x \in K} \frac{|R_{L,n}(c, s, x)|^p}{\sum_{k=-1}^{L} |R_{k,n}(c, s, x)|^p}, \tag{3.3}$$

where $\{R_{k,n}(c, s, x)\}_{k=0}^{\infty}$ denotes a perturbation of $\{R_k(c, s, x)\}_{k=0}^{\infty}$. More specifically, the recurrence coefficients for $\{R_{k,n}(c, s, x)\}_{k=0}^{\infty}$ are $\{a_{n+k}, b_{n+k}\}_{k=0}^{\infty}$, and the recurrence relation is given by

$$xR_{k,n}(c, s, x) = a_{k+1+n} R_{k+1,n}(c, s, x) + b_{k+n} R_{k,n}(c, s, x)$$
$$+ a_{k+n} R_{k-1,n}(c, s, x),$$
$$R_{-1,n}(c, s, x) = c, \qquad R_{0,n}(c, s, x) = s.$$

Since L is fixed, we have

$$\lim_{n \to \infty} \sup_{|c|^p+|s|^p=1, x \in K} |R_{k,n}(c, s, x) - R_k(c, s, x)| = 0$$

for $k = -1, 0, 1, 2, ..., L$, because all R_k's and $R_{k,n}$'s involved here are polynomials in c, s, and x, because they are generated through finitely many steps of three-term recursion, and because $a_{k+n} \to \frac{1}{2}$ and $b_{k+n} \to 0$ as $n \to \infty$. Noting that $\sum_{k=-1}^{L} |R_{k,n}(c, s, x)|^p \geqslant |c|^p + |s|^p = 1$, we obtain

$$\lim_{n \to \infty} \sup_{|c|^p+|s|^p=1, x \in K} \frac{|R_{L,n}(c, s, x)|^p}{\sum_{k=-1}^{L} |R_{k,n}(c, s, x)|^p}$$

$$= \sup_{|c|^p+|s|^p=1, x \in K} \frac{|R_L(c, s, x)|^p}{\sum_{k=-1}^{L} |R_k(c, s, x)|^p}.$$

Taking the superior limit of both sides of (3.3), and using the above equality, we obtain (3.1). ∎

In what follows we will prove that the right-hand side of (3.1) is bounded from above by const$(p)/L$, from which part (1) of Theorem 2.1 follows immediately, and part (2) follows with an unexpected continuity argument. To proceed we will need an auxiliary proposition.

LEMMA 3.2. *Let* $0<p<\infty$. *Then*

$$\sup_{\phi \in (0,2\pi),\, w \in \mathbb{C}} \frac{|1-we^{iL\phi}|^p}{\sum_{k=0}^{L} |1-we^{ik\phi}|^p} \leqslant \frac{4^{p+1}}{L} \tag{3.4}$$

holds for every positive integer L.

Proof. Fix $\phi \in (0, 2\pi)$ and $w \in \mathbb{C}$. Assume without loss of generality that $0<\phi \leqslant \pi$ and $|w| \leqslant 1$. Let $w = re^{i\psi}$ with $0 \leqslant r \leqslant 1$. Choose $0 \leqslant k_0 \leqslant L$ and $0<|t_0| \leqslant \pi$ such that

$$|1-re^{it_0}| = |1-re^{i(k_0\phi+\psi)}| = \max_{0 \leqslant k \leqslant L} |1-we^{ik\phi}|.$$

It suffices to show

$$\frac{|1-re^{it_0}|^p}{\sum_{k=0}^{L} |1-re^{i(k\phi+\psi)}|^p} \leqslant \frac{4^{p+1}}{L} \tag{3.5}$$

because (3.5) is stronger than (3.4). Define $A_1 = \{e^{it} : 4|t| < |t_0|\}$, which is an open arc containing 1 in the unit circle,. and $A_{-1} = \{e^{it} : |t_0| \leqslant 4|t| \leqslant 4\pi\}$, which is a closed arc containing -1. Clearly, A_1 and A_{-1} are disjoint, and their union is the whole unit circle. Also, the arc length of A_{-1} is not less than $3\pi/2$, and that of A_1 is not bigger than $\pi/2$ because $|t_0| \leqslant \pi$. We will show that among the points $e^{i(k\phi+\psi)}$, $k = 0, 1, ..., L$, there are substantially many of them lying on A_{-1}. To be more specific, let $E = \{k : 0 \leqslant k \leqslant L,\ e^{i(k\phi+\psi)} \in A_{-1}\}$. Then

$$|E| \geqslant \frac{L}{4}, \tag{3.6}$$

where $|\cdot|$ denotes the cardinality of a set. Once (3.6) is proved, we can show (3.5) quite easily. Indeed, for every $k \in E$ the inequality $|1-re^{i(k\phi+\psi)}|^2 = 1-2r\cos(k\phi+\psi)+r^2 \geqslant 1-2r\cos(t_0/4)+r^2 = |1-re^{it_0/4}|^2$ holds since $|\cos(k\phi+\psi)| \leqslant \cos(t_0/4)$ when $k \in E$. Thus,

$$\frac{|1-re^{it_0}|^p}{\sum_{k=0}^{L} |1-re^{i(k\phi+\psi)}|^p} \leqslant \frac{1}{|E|} \frac{|1-re^{it_0}|^p}{|1-re^{it_0/4}|^p}. \tag{3.7}$$

If for every fixed $|t_0| \leqslant \pi$, the function f is defined by

$$f(r) = \frac{|1-re^{it_0}|^2}{|1-re^{it_0/4}|^2} = \frac{1-2r\cos t_0 + r^2}{1-2r\cos(t_0/4)+r^2},$$

then the derivative of f is

$$f'(r) = \frac{2(1-r^2)(\cos(t_0/4)-\cos t_0)}{(1-2r\cos(t_0/4)+r^2)^2}.$$

Obviously, $f'(r) \geqslant 0$ when $0 \leqslant r \leqslant 1$. Therefore, for every fixed $|t_0| \leqslant \pi$, f is an increasing function of r in $[0, 1]$, which justifies the inequality

$$\frac{|1-re^{it_0}|}{|1-re^{it_0/4}|} \leqslant \frac{|1-e^{it_0}|}{|1-e^{it_0/4}|} = 4\cos\frac{t_0}{4}\cos\frac{t_0}{8} \leqslant 4. \tag{3.8}$$

From (3.6), (3.7), and (3.8) we obtain (3.5), which implies (3.4) by the definition of k_0 and t_0.

It remains to prove (3.6). Let us first address the case $k_0 \leqslant L/2$, and consider the equidistant points $e^{i(k\phi+\psi)}$, $k = k_0, k_0+1, ..., L$, on the unit circle. We will show that at least half of these points belong to A_{-1}, because the closed arc A_{-1} is longer than the open arc A_1. We use the mapping $t \mapsto e^{it}$ to wrap the unit circle counterclockwise with the interval (or rope) $[k_0\phi+\psi, (L+1)\phi+\psi)$. For each integer k in $\{k_0, k_0+1, ..., L\}$, call $k\phi+\psi$ a knot on the rope. The process starts with the first knot $\theta_0 = k_0\phi+\psi$, whose image $e^{i(k_0\phi+\psi)} = e^{it_0}$ is in A_{-1}. In the counterclockwise direction, first we cover the rest of A_{-1}, which contains the arc $\{e^{it}: -|t_0| \leqslant t \leqslant -|t_0|/4\}$, and then cover A_1. Stop wrapping at $e^{i|t_0|/4}$ just before we reach A_{-1} or at the end of the rope if the rope ends before we reach $e^{i|t_0|/4}$. Denote the portion of the rope we just used by $I_1 = [k_0\phi+\psi, \theta_1) = [\theta_0, \theta_1)$. It is clear that in this portion there are at least as many knots of I_1 wrapped to A_{-1} as to A_1, because the closed arc $\{e^{it}: -|t_0| \leqslant t \leqslant -|t_0|/4\}$ is longer than the open arc A_1 whose length is $|t_0|/2$, and the knots are equidistant. Now continue wrapping, cover A_{-1} and A_1, and again stop at the point $e^{i|t_0|/4}$ or at the end of the rope, whichever comes first. Let $I_2 = [\theta_1, \theta_2)$ be the portion we just used. We claim that in this second portion there are at least as many knots of I_2 wrapped to A_{-1} as to A_1. If there are no knots of I_2 wrapped to A_1, the claim is trivially true. So assume that there is at least one knot of I_2 wrapped to A_1. Since A_{-1} was wrapped first, its arc length is not less than $3\pi/2$, and the distance between consecutive knots in the rope is $\phi \leqslant \pi$, we conclude that there is at least one knot wrapped on A_{-1} before any knot on A_1. Now if ϕ is greater than the arc length of A_1, that is, $\phi \geqslant |t_0|/2$, then there can be at most one knot of I_2 wrapped to the open arc A_1, and the claim is correct. Otherwise, $0 < \phi < |t_0|/2 \leqslant \pi/2$. In this case the first knot of I_2 is actually wrapped to the upper half circle part of A_{-1} because $|t_0|/4 + \phi \leqslant \pi/4 + \pi/2 < \pi$. In the counterclockwise direction the rest of A_{-1} contains the closed arc $\{e^{it}: -|t_0| \leqslant t \leqslant -|t_0|/4\}$, and it is longer than the open arc A_1 whose length is $|t_0|/2$. Since the knots are equidistant, there must be at least as many knots of I_2 wrapped to A_{-1} as to A_1, and the claim is proved. Continue this process until the rope ends, and we find out that there are at least as many knots of the whole rope wrapped to A_{-1} as to A_1. This means that for at least half of the integer k's in $\{k_0, k_0+1, ..., L\}$, the point $e^{i(k\phi+\psi)}$ lies in A_{-1}, that is, $k \in E$. Therefore, $|E| \geqslant (L-k_0+1)/2$. Since $k_0 \leqslant L/2$, we also have $|E| \geqslant L/4$ which proves (3.6).

For the case $k_0 > L/2$, we consider $e^{i(k\phi+\psi)}$, $k=k_0, k_0-1, ..., 1$, and wrap the circle clockwise from $k_0\phi+\psi$ to $0\cdot\phi+\psi$. By repeating the argument above we know that at least half of the knots in $\{k\phi+\psi : k=k_0, k_0-1, ..., 1\}$ will be wrapped to A_{-1}, and again, $|E| \geqslant k_0/2 > L/4$. This completes the proof of (3.6).

As discussed earlier, (3.6) proves (3.5) and (3.4). ∎

Now we are ready for the

Proof of Theorem 2.1. As mentioned in Section 2, we can assume without loss of generality that either the recurrence coefficients $\{a_n, b_n\}_{n=0}^{\infty} \in CM(0, 1)$ or the measure $\alpha \in M(0, 1)$, and, therefore, we will prove (2.2) and (2.3) for $a=1$ and $b=0$.

Let $L \geqslant 1$ be a fixed integer, and let $\{R_k(c, s, x)\}_{k=0}^{\infty}$ be generated by (3.2). When $-1<x<1$, the characteristic equation of the second order difference equation (3.2) is $2xz = z^2+1$, and it has two distinct solutions $z_{\pm} = e^{\pm i\theta}$ with $\cos\theta = x$ and $0<\theta<\pi$. Thus, for $-1<x<1$, the solution to (3.2) is

$$R_k(c, s, x) = ue^{ik\theta} - ve^{-ik\theta}, \qquad k=-1, 0, 1, 2, ...,$$

where u and v depend on c, s, and x, but not on k. Since $|c|^p + |s|^p = 1$, we have $|u|+|v| \neq 0$. Hence,

$$|R_k(c, s, x)| = W|1-we^{ik\phi}|, \qquad k=-1, 0, 1, ...,$$

where either $\phi = 2\theta$ or $\phi = -2\theta$, and W and w depend on u and v. Therefore,

$$\frac{|R_L(c, s, x)|^p}{\sum_{k=-1}^{L} |R_k(c, s, x)|^p} = \frac{|1-we^{iL\phi}|^p}{\sum_{k=-1}^{L} |1-we^{ik\phi}|^p} \leqslant \frac{|1-we^{iL\phi}|^p}{\sum_{k=0}^{L} |1-we^{ik\phi}|^p}.$$

By Lemma 3.2,

$$\frac{|R_L(c, s, x)|^p}{\sum_{k=-1}^{L} |R_k(c, s, x)|^p} \leqslant \frac{4^{p+1}}{L} \tag{3.9}$$

holds for $-1<x<1$. The above is actually true for $-1 \leqslant x \leqslant 1$ when $|c|^p + |s|^p = 1$ because of continuity.

For part (1) of Theorem 2.1, we assume $\{a_n, b_n\}_{n=0}^{\infty} \in CM(0, 1)$ and fix an integer $L \geqslant 1$. Then (3.1) holds with $K = [-1, 1]$, and (2.2) follows from (3.1) and (3.9) because L can be arbitrarily large.

For part (2), again we fix $L \geqslant 1$. First we note that $|R_L(c, s, x)|^p / \sum_{k=-1}^{L} |R_k(c, s, x)|^p$ is uniformly continuous in c, s, and x when

$|c|^p + |s|^p = 1$ and $x \in [-2, 2]$ because $\{R_k(c, s, x)\}_{k=0}^{\infty}$ are polynomials in c, s, and x, and the denominator is not less than 1. Hence, for the fixed L, there is a $0 < \delta \leqslant 1$ such that

$$\left| \frac{|R_L(c, s, x)|^p}{\sum_{k=-1}^{L} |R_k(c, s, x)|^p} - \frac{|R_L(c', s', x')|}{\sum_{k=-1}^{L} |R_k(c', s', x')|^p} \right| \leqslant \frac{1}{L}, \qquad (3.10)$$

whenever $x, x' \in [-2, 2]$, $|c|^p + |s|^p = 1$, $|c'|^p + |s'|^p = 1$, and $|c - c'| + |s - s'| + |x - x'| \leqslant \delta$. Combining (3.9) and (3.10) we obtain

$$\sup_{|c|^p + |s|^p = 1, x \in [-1-\delta, 1+\delta]} \frac{|R_L(c, s, x)|^p}{\sum_{k=-1}^{L} |R_k(c, s, x)|^p} \leqslant \frac{1 + 4^{p+1}}{L} \leqslant \frac{4^{p+2}}{L}. \qquad (3.11)$$

Now let the measure $\alpha \in M(0, 1)$. Then $\{a_n(\alpha), b_n(\alpha)\}_{n=0}^{\infty} \in CM(0, 1)$, and Lemma 3.1 holds for the associated orthogonal polynomials $\{p_k\}_{k=0}^{\infty}$ with $K = [-1-\delta, 1+\delta]$, and $Z = \varnothing$ because $p_0(x) = \text{const} > 0$. Following (3.1) and (3.11), we have

$$\overline{\lim_{n \to \infty}} \sup_{x \in [-1-\delta, 1+\delta]} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \leqslant \frac{4^{p+2}}{L}. \qquad (3.12)$$

Recall that Blumenthal's theorem (cf. Section 1 and [1; 10, p. 23]) claims that if the measure $\alpha \in M(0, 1)$, then $\operatorname{supp}(\alpha) \backslash [-1-\delta, 1+\delta]$ consists of at most finitely many mass-points of α. At each such point x, we have $\sum_{k=0}^{\infty} |p_k(x)|^2 < \infty$ (cf. Section 1), thus, $p_n(x) \to 0$ as $n \to \infty$. This implies that

$$\overline{\lim_{n \to \infty}} \sup_{x \in \operatorname{supp}(\alpha) \backslash [-1-\delta, 1+\delta]} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} = 0. \qquad (3.13)$$

From (3.12) and (3.13),

$$\overline{\lim_{n \to \infty}} \sup_{x \in \operatorname{supp}(\alpha)} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \leqslant \frac{4^{p+2}}{L}.$$

Finally, letting $L \to \infty$, we obtain

$$\lim_{n \to \infty} \sup_{x \in \operatorname{supp}(\alpha)} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} = 0.$$

We point out that in the denominator the upper limit of the summation is n instead of $n-1$. But with the following identity,

$$\frac{|p_n(x)|^p}{\sum_{k=0}^{n-1} |p_k(x)|^p} = \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \Bigg/ \left[1 - \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \right],$$

we see that n and $n-1$ makes no difference in the limit relation. This finishes the proof of part (2) of Theorem 2.1 for $\alpha \in M(0, 1)$. A similar remark applies for part (1) as well. ∎

The next lemma will be used to prove Theorem 2.2.

LEMMA 3.3. *Let $\{d_n \neq 0\}_{n=0}^{\infty}$ be sequence of complex numbers depending on some parameter set Ξ, and let $|\rho| > 1$ hold uniformly in Ξ. Then*

$$\lim_{n \to \infty} \frac{d_{n+1}}{d_n} = \rho \qquad \textit{uniformly in } \Xi$$

$$\Leftrightarrow \lim_{n \to \infty} \frac{1}{d_n} \sum_{k=0}^{n-1} d_k = \frac{1}{\rho - 1} \qquad \textit{uniformly in } \Xi.$$

The proof is elementary, and as such, we leave it to the reader.

Proof of Theorem 2.2. For $p=2$ the "$\Rightarrow$" part of the theorem was proved in [10, p. 31, Theorem 4.1.11, see λ_n^*]. If $0 < p < \infty$ is arbitrary, then we proceed as follows. Poincaré's theorem [13] on the asymptotic behavior of the ratios f_{n+1}/f_n of solutions $\{f_n\}$ of linear difference equations with variable coefficients was generalized for orthogonal polynomials in [10, p. 33, Theorem 4.1.13]. This generalization consists of

$$\lim_{n \to \infty} \frac{p_{n+1}(\alpha, x)}{p_n(\alpha, x)} = \frac{x-b}{a} + \sqrt{\left(\frac{x-b}{a}\right)^2 - 1}, \tag{3.14}$$

which holds uniformly on every compact set $K \subset \mathbb{C} \backslash \operatorname{supp}(\alpha)$, whenever $\alpha \in M(b, a)$ (for an alternative proof see [9, Theorem 3 and formula (16)]). In view of (3.14), we can apply Lemma 3.3 with $d_k = |p_k(\alpha, x)|^p$ to obtain (2.5), and the limit in (2.5) is uniform on K.

Conversely, if formula (2.5) holds uniformly on compact subsets of $\mathbb{C} \backslash \operatorname{supp}(\alpha)$, then by Lemma 3.3,

$$\lim_{n \to \infty} \left| \frac{p_{n+1}(\alpha, x)}{p_n(\alpha, x)} \right| = \left| \frac{x-b}{a} + \sqrt{\left(\frac{x-b}{a}\right)^2 - 1} \right|. \tag{3.15}$$

When x is real and it is located to the right of $\operatorname{supp}(\alpha)$, then $p_k(x) = |p_k(x)|$ for all $k = 0, 1, \ldots$, and, hence, (3.14) holds as well. Therefore, by [10, p. 32, Theorem 4.1.12], we have $\alpha \in M(b, a)$.[3]

We remark that when p is a positive integer, the proof remains valid with all the absolute value signs removed, and this takes care of the second part of Theorem 2.2. ∎

[3] As pointed out earlier, a and b play opposite roles in [10] than in the present paper.

4. Applications

In what follows

$$\lambda_n(\alpha, \cdot) = \left[\sum_{k=0}^{n-1} p_k^2(\alpha, \cdot) \right]^{-1} \tag{4.0.1}$$

is the nth Christoffel function associated with the measure α. It was shown in [10] that for $\alpha \in M(b, a)$ with $a > 0$ and $b \in \mathbb{R}$, the limit relation

$$\lim_{n \to \infty} \lambda_n(\alpha, x)\, p_n^2(\alpha, x) = 0, \qquad x \in [b-a, b+a], \tag{4.0.2}$$

has a number of applications such as estimating the growth of orthogonal polynomials and Lebesgue functions of orthogonal series expansions, comparative asymptotics for Christoffel functions, and so forth. Unfortunately, in [10, p. 11, Theorem 3.1.9] this limit was proved only locally uniformly in $(b-a, b+a)$. In view of our extension of uniform convergence to the entire interval $[b-a, b+a]$ now we are able to improve many of the applications of (4.0.2).

4.1. *Orthogonal Polynomials of the Erdős and Al. Magnus Classes*

Corollary 4.1.1. *Let $I \subset \mathbb{R}$ be a compact interval. If α is an Erdős measure on I, that is,* $\operatorname{supp}(\alpha) = I$ *and* $\alpha'(x) > 0$ *a.e. in I, then the corresponding orthogonal polynomials $\{p_n(\alpha)\}_{n=0}^{\infty}$ satisfy*

$$\lim_{n \to \infty} \max_{x \in I} \lambda_n(\alpha, x)\, p_n^2(\alpha, x) = 0. \tag{4.1.1}$$

Proof. Let $I = [b-a, b+a]$. Since α is an Erdős measure, by a theorem of Rahmanov (cf. [14, 7]) we have $\lim_{n \to \infty} a_n = a/2$ and $\lim_{n \to \infty} b_n = b$, that is, $\alpha \in M(b, a)$. Taking $p = 2$ in Theorem 2.1 and noting that $Z = \varnothing$, we immediately obtain (4.1.1). ∎

Next we apply a theorem of Alphonse Magnus [6] to obtain a uniform convergence result for certain complex weights.

Corollary 4.1.2. *Let $I \subset \mathbb{R}$ be a compact interval. Let w be a Magnus weight in I, that is, $w = gw$, where g is a non-vanishing complex valued continuous function in I and $\omega \in L_1(I)$ is positive a.e. in I. Let f be the Stieltjes transform of w, that is,*

$$f(z) = \int_I \frac{w(t)\, dt}{z - t}, \qquad z \notin I.$$

Then the denominators $\{P_n\}$ *of the diagonal Padé approximants* $\{Q_n/P_n\}$ *of* f *about* ∞ *satisfy*

$$\lim_{n\to\infty} \max_{x\in I} \frac{|P_n(x)|^p}{\sum_{k=0}^{n-1} |P_k(x)|^p} = 0 \tag{4.1.2}$$

for all $0 < p < \infty$.

Proof. By [6, p. 34, Theorem 6.1], there is an integer N such that

(1) P_k is a polynomial of precise degree k for $k > N$,

(2) $\int_I P_m P_n \, d\alpha = \delta_{nm}$, $m > N$, $n > N$,

(3) the three-term recurrence relation

$$xP_k(x) = a_{k+1} P_{k+1}(x) + b_k P_k(x) + a_k P_{k-1}(x)$$

holds for all $k \geqslant N$, and

(4) $\lim_{k\to\infty} a_k = a/2$ and $\lim_{k\to\infty} b_k = b$, where $I = [b-a, b+a]$.

To apply Theorem 2.1 we just need a shift in the subscript. Note that $p_k = P_{N+k}$ $(k \geqslant -1)$ satisfies the three-term recurrence (1.1) and the recurrence coefficients have the required convergence property (2.1). By part (1) of Theorem 2.1,

$$\lim_{n\to\infty} \max_{x\in I} \frac{|P_n(x)|^p}{\sum_{k=N}^{n-1} |P_k(x)|^p} = 0, \qquad 0 < p < \infty,$$

where it is understood that all common factors of $P_{N-1}(x)$ and $P_n(x)$ have been cancelled in the division. Finally, replacing the lower limit in the sum from $k = N$ to $k = 0$ we obtain (4.1.2). ∎

Alphonse Magnus pointed out in a private communication to us that it should be possible to prove that for n large enough P_n and P_{n-1} have no common zeros, and so probably there is no need for any cancellation.[4]

4.2. *Estimates of Orthogonal Polynomials and Lebesgue Functions*

The following lemma immediately follows from the extremal property of Christoffel functions (cf. [3, p. 25, Theorem 1.4.1, and p. 105, Theorem 3.3.4]).

LEMMA 4.2.1. *Let* $a > 0$ *and* $b \in \mathbb{R}$. *If* α *satisfies*

$$\alpha'(x) \geqslant C[a^2 - (x-b)^2]^{-1/2}, \qquad x \in [a-b, a+b], \tag{4.2.1}$$

[4] See the proof of (2) in Theorem 2.1 which is right after the statement of Theorem 2.1 in Section 2.

then

$$\lambda_n^{-1}(\alpha, x) \leqslant \frac{2n}{\pi C}, \qquad x \in [a-b, a+b]. \tag{4.2.2}$$

From Corollary 4.1.1 and Lemma 4.2.1 we obtain the following pointwise estimate for orthogonal polynomials.

COROLLARY 4.2.2. *Let* $a>0$ *and* $b \in \mathbb{R}$. *Let either* $\operatorname{supp}(\alpha) = [b-a, b+a]$ *or* $\alpha \in M(b, a)$, *and assume that* α *satisfies* (4.2.1). *Then*

$$p_n(\alpha, x) = o(\sqrt{n}), \qquad n = 1, 2, ..., \tag{4.2.3}$$

uniformly for $x \in [b-a, b+a]$.

The estimate (4.2.3) is placed in proper prospective if we note that after Rahmanov [15] disproved Steklov's conjecture on the uniform boundedness of orthogonal polynomials whose weight functions satisfy $w(x) \geqslant \text{const}(1-x^2)^{-1/2}$ for $x \in [-1, 1]$, he also proved in [16] that in general for such weights, even the estimate $p_n(x) = O(n^\varepsilon)$ in $[-1, 1]$ fails for some $\varepsilon < \frac{1}{2}$. We also point out that it is mistakenly claimed [15, p. 594] that the unit circle analogue of (4.2.3) was proved in [4, p. 40, Theorem 3.5] (cf. *Remark* in Section 2).

Our next result is about estimating the Lebesgue functions (that is, norms)

$$\Omega_n(\alpha, x) = \sup_{\|f\|_\infty \leqslant 1} |S_n(\alpha, f, x)|$$

for partial sums

$$S_n(\alpha, f) = \sum_{k=0}^{n-1} c_k(\alpha, f)\, p_k(\alpha)$$

of Fourier expansions in orthogonal polynomials, where the Fourier coefficients are given by $c_k(\alpha, f) = \int_{\mathbb{R}} f p_k(\alpha)\, d\alpha$.

In what follows, we define the *function* $\hat{\alpha}$ by $\hat{\alpha}(t) = \int_{(-\infty, t)} d\alpha$.

COROLLARY 4.2.3. *Let* $a>0$ *and* $b \in \mathbb{R}$. *Let* $a \in M(b, a)$ (*cf.* (1.5)), *and let the* function $\hat{\alpha}$ *be uniformly continuous on a set* $\Delta \subseteq [b-a, b+a]$. *Then*

$$\lim_{n \to \infty} \sup_{x \in \Delta} \lambda_n(\alpha, x)\, \Omega_n^2(\alpha, x) = 0. \tag{4.2.4}$$

If, in addition, α *satisfies* (4.2.1) *then*

$$\Omega_n(\alpha, x) = o(\sqrt{n}), \qquad n = 1, 2, ..., \tag{4.2.5}$$

uniformly for $x \in \Delta$.

Proof. From [12, p. 32 formula (4.14.8)].

$$\lambda_n(\alpha, x)\,\Omega_n^2(\alpha, x) \leqslant 2[\hat{\alpha}(x+\varepsilon) - \hat{\alpha}(x-\varepsilon)] + 4a_n^2(\alpha)\,\varepsilon^{-2}\lambda_n(\alpha, x)$$
$$\times [p_n^2(\alpha, x) + p_{n-1}^2(\alpha, x)] \int_{\mathbb{R}} d\alpha$$

for every $\varepsilon > 0$. Applying Theorem 2.1 (cf. (4.0.1)), we obtain

$$\overline{\lim_{n\to\infty}} \sup_{x\in\Delta} \lambda_n(\alpha, x)\,\Omega_n^2(\alpha, x) \leqslant 2 \sup_{x\in\Delta} [\hat{\alpha}(x+\varepsilon) - \hat{\alpha}(x-\varepsilon)]$$

for every $\varepsilon > 0$. Since the *function* $\hat{\alpha}$ is uniformly continuous on Δ, we have

$$\overline{\lim_{n\to\infty}} \sup_{x\in\Delta} \lambda_n(\alpha, x)\,\Omega_n^2(\alpha, x) = 0.$$

Finally, (4.2.5) follows form (4.2.4) and (4.2.2). ∎

4.3. *Uniform Convergence of a Sequence of Positive Operators*

The so called G operator was introduced in [10, p. 74, Sect. 6.2] (see also [8, p. 53; 12, p. 19]), and it is defined by

$$G_n(\alpha, f, x) = \lambda_n(\alpha, x) \int_{\mathbb{R}} f(t)\, K_n^2(\alpha, x, t)\, d\alpha(t), \tag{4.3.1}$$

where $\lambda_n(\alpha)$ is the nth Christoffel function (cf. (4.0.1)), and $K_n(\alpha)$ is the nth reproducing kernel, that is,

$$K_n(\alpha, x, t) = \sum_{k=0}^{n-1} p_k(\alpha, x)\, p_k(\alpha, t).$$

By the Christoffel–Darboux formula [17, p. 43]

$$K_n(\alpha, x, t) = a_n(\alpha) \frac{p_n(\alpha, x)\, p_{n-1}(\alpha, t) - p_n(\alpha, t)\, p_{n-1}(\alpha, x)}{x - t}. \tag{4.3.2}$$

COROLLARY 4.3.1. *Let $a > 0$ and $b \in \mathbb{R}$. If $\alpha \in M(b, a)$ and $f \in L_\infty(\mathbb{R})$ is uniformly continuous on a set $\Delta \subseteq [b-a, b+a]$, then*

$$\lim_{n\to\infty} \sup_{x\in\Delta} |G_n(\alpha, f, x) - f(x)| = 0. \tag{4.3.3}$$

Proof. We could use Korovkin's theorem to prove this but a direct proof is equally simple. It follows from the definition of the kernel function that

$$\lambda_n(\alpha, x) \int_{\mathbb{R}} K_n^2(\alpha, x, t)\, d\alpha(t) = 1. \tag{4.3.4}$$

Hence, for $f=1$ we have $G_n(\alpha, f)=f$. Therefore, for a fixed $\delta>0$

$$|G_n(\alpha, f, x)-f(x)| \leqslant \lambda_n(\alpha, x) \int_{|x-t| \leqslant \delta} |f(t)-f(x)|\, K_n^2(\alpha, x, t)\, d\alpha(t)$$
$$+\lambda_n(\alpha, x) \int_{|x-t|>\delta} |f(t)-f(x)|\, K_n^2(\alpha, x, t)\, d\alpha(t).$$

By (4.3.4) we have

$$\lambda_n(\alpha, x) \int_{|x-t| \leqslant \delta} |f(t)-f(x)|\, K_n^2(\alpha, x, t)\, d\alpha(t) \leqslant \omega(f, \delta)$$

for all $x \in \Delta$, where $\omega(f, \delta)=\sup_{|x-t| \leqslant \delta, x \in \Delta} |f(x)-f(t)|$ is the modulus of continuity of f. Moreover, by the Christoffel–Darboux formula (4.3.2),

$$\lambda_n(\alpha, x) \int_{|x-t|>\delta} |f(t)-f(x)|\, K_n^2(\alpha, x, t)\, d\alpha(t)$$
$$\leqslant 2\, \|a(\alpha)\|_\infty\, \|f\|_\infty\, \delta^{-2} \lambda_n(\alpha, x)(p_n^2(\alpha, x)+p_{n-1}^2(\alpha, x))$$

for all $x \in \Delta$, where $\|a(\alpha)\|_\infty=\sup_{n \geqslant 0} |a_n(\alpha)|<\infty$ (cf. (1.4)). Combining the last three inequalities, we obtain

$$|G_n(\alpha, f, x)-f(x)|$$
$$\leqslant \omega(f, \delta)+2\, \|a(\alpha)\|_\infty\, \|f\|_\infty\, \delta^{-2} \lambda_n(\alpha, x)(p_n^2(\alpha, x)+p_{n-1}^2(\alpha, x))$$

for all $x \in \Delta$. Since $\alpha \in M(b, a)$, by Theorem 2.1 we have

$$\overline{\lim_{n \rightarrow \infty}}\, \sup_{x \in \Delta} |G_n(\alpha, f, x)-f(x)| \leqslant \omega(f, \delta)$$

for every $\delta>0$. Since f is uniformly continuous in Δ, $\omega(f, \delta) \rightarrow 0$ as $\delta \rightarrow 0$, which proves (4.3.3). ∎

4.4. *Christoffel Functions and Cotes Numbers*

COROLLARY 4.4.1. *If $\alpha \in M(b, a)$ with $a>0$ and $b \in \mathbb{R}$, then for every fixed integer m*

$$\lim_{n \rightarrow \infty} \max_{x \in [b-a, b+a]} \left|\frac{\lambda_{n+m}(\alpha, x)}{\lambda_n(\alpha, x)}-1\right|=0.$$

Proof. First let m be negative. Then it follows from the definition of Christoffel functions (4.0.1) that

$$1 \geqslant \frac{\lambda_{n+m}(\alpha, x)}{\lambda_n(\alpha, x)}=1-\sum_{k=n+m}^{n-1} p_k^2(\alpha, x)\, \lambda_{n+m}(\alpha, x)$$
$$\geqslant 1-\sum_{k=n+m}^{n-1} p_k^2(\alpha, x)\, \lambda_k(\alpha, x).$$

Since m is fixed and each term in the last sum converges to zero uniformly in $[b-a, b+a]$ (cf. Theorem 2.1), and this proves Corollary 4.4.1 for negative m's, whereas for positive m's we can simply take reciprocals. ∎

Given a measure α and a function g, we define the new measure α_g by

$$d\alpha_g = g\, d\alpha.$$

The following result gives comparative asymptotics of the two Christoffel functions $\lambda_n(\alpha)$ and $\lambda_n(\alpha_g)$.

COROLLARY 4.4.2. *Let $\alpha \in M(b, a)$ with $a > 0$ and $b \in \mathbb{R}$, and let $g \geq 0$ on $\mathbb{R}$. Assume that there is a polynomial Π_m such that $\Pi_m g$ and $\Pi_m g^{-1}$ are both uniformly continuous in a set $\Delta \subseteq [b-a,\ b+a]$, and they are both uniformly bounded in $\mathbb{R}$. Then*

$$\lim_{n \to \infty} \frac{\lambda_n(\alpha_g, x)}{\lambda_n(\alpha, x)} = g(x)$$

uniformly in every compact subset of Δ void of zeros of Π_m. In particular, if g is bounded and strictly positive in $\mathbb{R}$, and it is continuous in $[b-a, b+a]$, then the convergence is uniform in the entire interval $[b-a, b+a]$.

For the partial case when Δ is either a single point in $[b-a, b+a]$ or $\Delta \subset (b-a,\ b+a)$ is a closed interval, this corollary was proved in [10, p. 78, Theorem 6.2.6]. Before the proof, we point out that Corollary 4.4.2 enables one to find the asymptotic values of the Cotes numbers in the Gauss–Jacobi quadrature process (cf. [17, p. 47])

$$\int_{\mathbb{R}} f(x)\, d\alpha(x) = \sum_{k=1}^{n} \lambda_{kn}(\alpha)\, f(x_{kn}(\alpha)),$$

where $\{x_{kn}(\alpha)\}_{k=1}^{n}$ are the zeros of the orthogonal polynomials $p_n(\alpha)$, and the Cotes numbers $\lambda_{kn}(\alpha)$ are given by $\lambda_{kn}(\alpha) = \lambda_n(\alpha, x_{kn}(\alpha))$. For instance, if α is the Chebyshev measure, that is, $d\alpha(x) = (1-x^2)^{-1/2}\, dx$ in $[-1, 1]$ then the Christoffel functions $\lambda_n(\alpha)$ can explicitly be computed (cf. [10, p. 79, Example 6.2.8]). For the Cotes numbers associated with the measure α_g, we have

$$\lambda_n(\alpha_g, x_{kn}) \approx 2\pi \frac{g(x_{kn})}{2n - 1 + U_{2n-2}(x_{kn})}$$

uniformly for $n = 1, 2, \ldots,$ and $k = 1, 2, \ldots, n$, where U_m is the second kind Chebyshev polynomial of degree m, that is, $U_m(x) = \sin((m+1)\theta)/\sin\theta$, $x = \cos\theta$.

Proof of Corollary 4.4.2. Pick a compact subset of Δ void of zeros of Π_m, say, K. As in [10, p. 77, Theorem 6.2.5] or [12, pp. 20–21, Theorem 4.5.8] we have

$$\frac{\lambda_{n+m}(\alpha, x)}{\lambda_n(\alpha_g, x)} \leqslant \frac{1}{\Pi_m^2(x)} G_{n+m}\left(\alpha, \frac{\Pi_m^2}{g}, x\right) \tag{4.4.1}$$

and

$$\frac{\lambda_n(a_g, x)}{\lambda_{n-m}(\alpha, x)} \leqslant \frac{1}{\Pi_m^2(x)} G_{n-m}(\alpha, \Pi_m^2 g, x). \tag{4.4.2}$$

Since $\Pi_m^2 g^{\pm 1}$ and α satisfy the conditions of Corollary 4.3.1 we obtain

$$\lim_{n\to\infty} \frac{1}{\Pi_m^2(x)} G_{n+m}\left(\alpha, \frac{\Pi_m^2}{g}, x\right) = \frac{1}{g(x)}$$

and

$$\lim_{n\to\infty} \frac{1}{\Pi_m^2(x)} G_{n-m}(\alpha, \Pi_m^2 g, x) = g(x)$$

uniformly for $x \in K$. Applying Corollary 4.4.1 to the left sides of inequalities (4.4.1) and (4.4.2) we obtain

$$\varlimsup_{n\to\infty} \frac{\lambda_n(\alpha, x)}{\lambda_n(\alpha_g, x)} \leqslant \frac{1}{g(x)} \qquad \text{and} \qquad \varlimsup_{n\to\infty} \frac{\lambda_n(\alpha_g, x)}{\lambda_n(\alpha, x)} \leqslant g(x)$$

uniformly for $x \in K$. This completes the proof. ∎

Another application of Corollary 4.3.1 is related to generalized Christoffel functions $\Lambda_n(\alpha)$ which is defined by

$$\Lambda_n(\alpha, N, h, X) = \min_{\deg \Pi < n,\, \Pi(X_j) = h(X_j)} \int_{\mathbb{R}} |\Pi(t)|^2 \, d\alpha(t),$$

where N is a fixed positive integer, $X = \{X_j\}_{j=1}^N \in \mathbb{R}^N$, and $h: \mathbb{R} \to \mathbb{C}$ (cf. [11]). For $N = 1$ we have $\Lambda_n(\alpha, N, h, X) = |h(X)|^2 \lambda_n(\alpha, X)$, where $\lambda_n(\alpha)$ is the regular Christoffel function (4.0.1) (cf. [3, p. 25, Theorem 1.4.1]).

For the class $M(b, a)$, generalized Christoffel functions were investigated in [11], and a careful analysis of the proof of [11, p. 302, Lemma] shows that Theorem 2.1 and Corollary 4.3.1 yield

COROLLARY 4.4.3. *Let $\alpha \in M(b, a)$ with $a > 0$ and $b \in \mathbb{R}$, let N be a positive integer, and let $h: \mathbb{R} \to \mathbb{C}$ be bounded. Then for all fixed $r > 0$*

$$\lim_{n\to\infty} \frac{\Lambda_n(\alpha, N, h, X)}{\sum_{j=1}^N |h(X_j)|^2 \lambda_n(\alpha, X_j)} = 1$$

uniformly for all $X = \{X_j\}_{j=1}^N \in [b-a, b+a]^N$ such that $|X_i - X_j| \geqslant r\,\delta_{ij}$.

5. Epilogue

In this paper we considered the relative growth of orthogonal polynomials associated with measures in $M(b, a)$, and related second order recurrences of the form (1.1). It is natural to ask what happens to more general measures, and to higher order linear difference equations with variable polynomial coefficients. As opposed to the case treated in this paper, the latter one seems to be much more delicate. Very recently, this was studied from a more general point of view in [5], where "sub-exponential" growth of solutions of certain operator difference equations in normed spaces was proved. We expect that uniform convergence results similar to Theorem 2.1 may be proved by combining the methods in this paper with those in [5]. Poincaré's theorem [13] can still be applied to investigate the asymptotic behavior of the ratios f_{n+1}/f_n of solutions $\{f_n\}$ of such higher order linear difference equations.

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